THE MATRICULATION EXAMINATION BOARD — 27.9.2000 — MATHEMATICS ADVANCED COURSE

At most ten problems may be answered. There are several items [marked **a)**, **b)**, etc.] in some of the problems; all of them are to be considered for a complete answer.

1. Simplify the following expressions:

$$\textbf{a)}\quad (x^{n-1})^{n-1}\cdot(x^n)^{2-n}, \qquad \textbf{b)}\quad \sqrt[3]{a}(\sqrt[3]{a^2}-\sqrt[3]{a^5}).$$

2. Solve the equation

$$\sqrt{x-2}=1+\frac{2}{\sqrt{x-2}}.$$

3. A journey is travelled at a steady speed. When there is still 40% of the journey to be travelled, the speed is increased by 20%. Find the percentage decrease of the time used for the whole journey.

4. The top of a tower can be seen from a place at an angle of 3.5 degrees with respect to the horizontal plane. From another place exactly half a kilometer farther away the angle is 2.5 degrees. Find the height of the tower and the distances of the places from the tower. The ground is assumed to be flat.

5. Prove that $xy+yz+zx=-\frac{1}{2}$, if $x+y+z=0$ and $x^2+y^2+z^2=1$.

6. The polynomial $p(x)$ of degree 3 has a double zero at $x=2$; further, $p(3)=15$ and $p'(1)=0$. Find $p(x)$.

7. A bulb of a tulip variety will begin to grow with a probability of 0.7. What is the minimum number of bulbs that must be planted in order to get at least two tulips with a probability greater than 99%?

8. The vectors $\bar{a}$, $\bar{b}$ and $\bar{c}$ are the position vectors of the vertices of a triangle. These vectors and the position vector $\bar{p}$ of a point P satisfy the equations

$$\begin{cases}(\bar{p}-\bar{a})\cdot(\bar{b}-\bar{c})=0,\\(\bar{p}-\bar{b})\cdot(\bar{c}-\bar{a})=0.\end{cases}$$

How can the point P be characterized? Prove also that

$$(\bar{p}-\bar{c})\cdot(\bar{a}-\bar{b})=0.$$

This is a vector-algebra-based proof for a theorem concerning a triangle from elementary geometry. What is the theorem?

9. The diameter AB of a circular swamp is 1 km. An orienteerer likes to go from A to B as fast as possible. How must he choose his path, if he runs 10 km/h on the firm ground and 5 km/h in the swamp?

10. Prove by induction that $\dfrac{n^3+5n}{6}$ is a whole number, when n is a whole number and $n \geq 1$.

11. The equations of the lines s_1 and s_2 are $3x - 4y - 4 = 0$ and $x - 2y + 2 = 0$, respectively. One person travels from the point $P_0 = (0, -1)$ on the line s_1 in the direction of the positive y-axis, until the line s_2 is reached; then, he continues in the direction of the positive x-axis until the line s_1 is again reached at a point P_1. The step is repeated starting from the point P_1, and a point P_2 on the line s_1 is found in a similar way, etc. Compute the length of the stair-formed path to be travelled in this way from the point P_0 to the point P_n. What is the limit of the length, when $n \to \infty$?

12. Investigate which points (x, y) in the plane satisfy the equation $\log_y x = \log_x y$. Sketch a figure.

13. Function $f : \mathbb{R} \to \mathbb{R}$ is defined as follows:

$$f(x) = \int_x^{3x} \sqrt{t^2+1}\, dt.$$

Find the derivative $f'(x)$ and investigate the extreme values of the function f.

14. The slope of the tangent line of a curve at the point (x, y) is half of the slope of the line through this point and origin. Find the equation of the curve, when it is known, additionally, that the point $(4, 1)$ lies on the curve.

15. Find all solutions for the Diophantine equation $10x + 4y = 36$.