THE MATRICULATION EXAMINATION BOARD — 22.3.2000 — MATHEMATICS ADVANCED COURSE

At most ten problems may be answered. There are several items [marked **a)**, **b)**, etc.] in some of the problems; all of them are to be considered for a complete answer.

1. The perimeter of an isosceles triangle is 15 cm. Its equal sides are 1.5 cm longer than the base. Find the area of the triangle.

2. The jazz band of an upper secondary school gave a concert. The profit of 192 euro (€) from the concert is to be shared equally among the members of the band. If there had been two members more, everyone would have got 8 € less. How many members were there in the band?

3. **a)** Simplify the expression $\dfrac{\frac{1}{x}-x}{\frac{1}{x}+1}$. **b)** Simplify the expression $\dfrac{x-1}{(1-\frac{1}{\sqrt{x}})(1+\frac{1}{\sqrt{x}})}$.

c) Then solve the equation $\dfrac{\frac{1}{x}-x}{\frac{1}{x}+1} = \dfrac{x-1}{(1-\frac{1}{\sqrt{x}})(1+\frac{1}{\sqrt{x}})}$.

4. The water content of fresh apples is 80% and the sugar content 4%. Find the sugar content of the same apples expressed as a percentage, when they are dried in such a way that the water content is 20%.

5. Two mobile phone masts can be seen from a place at a distance of 5.27 km from one of the masts and 3.16 km from the other. The angle between the sight lines to the masts is $72^\circ\,50'$. Find the distance between the masts. All distances are measured horizontally and the possible height differences of the terrain are not to be taken into account.

6. The vertex of an angle is at the point $(1,2)$ and the points $(4,6)$, $(13,-3)$ are located on its sides. Compute the unit vector parallel to the bisector of the angle.

7. A function f is defined in the set of real numbers so that the slope of the tangent line of the graph of the function at an arbitrary point (x,y) is $k(x) = 1 - e^{-2x}$. The minimum value of the function f is 2. Find f.

8. The number of incoming calls to a telephone exchange is distributed according to the *Poisson distribution*: probability of n (≥ 0) incoming calls in one minute is $p_n = \frac{a^n}{n!}e^{-a}$, where the constant a describes the intensity of the phone traffic. Compute the probability of having 5 or more incoming calls in a minute, when $a = 3$.

9. The *central projection* of a point P to the line s with the centre of projection at the point K is defined to be the intersection point of the line s and the line through K and P (if the intersection exists). Let the centre of projection be at the point $K = (1,4)$ and the line s be the x-axis. Let $A = (0,1)$, $B = (4,3)$. Investigate which points on the x-axis are the projections of the points of the line segment AB in this central projection. Which point is the projection of the midpoint of the segment AB? If equidistant points are set on the line segment AB, are the corresponding projection points equidistant on the x-axis?

10. A similar right pyramid with four sides is set on each face of a cube. The common height of the pyramids is determined in such a way that the top vertices of adjacent pyramids and the common edge line of the corresponding faces are coplanar. Then, the cube and the pyramids form a polyhedron called a *rhombic dodecahedron*. Its faces are parallelograms formed of two faces of adjacent pyramids. **a)** Find the ratio of the height of the pyramid to the length of the edge of the cube. **b)** Find the angles of the face of the rhombic dodecahedron with an accuracy of one degree. **c)** Find the ratio of the volume of the rhombic dodecahedron to the volume of the cube.

11. A vessel has the form of an open circular cone standing on its vertex. The bottom radius of the cone is 6.6 cm and the slant height 11.0 cm. The vessel is full of water. A ball is set in the vessel in such a way that it is tangent to the surface of the cone. Find the radius of the ball when the quantity of water running out of the vessel is as large as possible.

12. Are the following propositions true? **a)** If the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ is positive for all values of the variable, the function increases infinitely, i.e. $\lim_{x\to\infty} f(x) = \infty$. **b)** If the derivative of the function $f : \mathbb{R} \to \mathbb{R}$ is greater than a positive constant for all values of the variable, then $\lim_{x\to\infty} f(x) = \infty$. Justify your answer.

13. Calculate the derivative of the function f, $f(x) = e^{-x}(\sin x + \cos x)$. Let A_0, A_1, A_2, $\ldots$ be the regions bounded by the curve $y = e^{-x}\sin x$ and x-axis, when $x \geq 0$. Prove that the areas of A_0, A_1, A_2, $\ldots$ form a geometric sequence. Evaluate the integral $\int_0^\infty |e^{-x}\sin x|\,dx$.

14. The number of mosquitoes at a summer event was 200 at the beginning of the gathering and three hours later 700. The rate of growth of the number at time t was proportional to the number of mosquitoes at that moment. Write the differential equation describing the amount of mosquitoes and find the the amount of mosquitoes at an arbitrary time t by solving the equation. What was the number of mosquitoes five hours after the beginning of the event?

15. Use the Euclidean algorithm for finding the greatest common divisor (highest common factor) $\gcd(34086, 14630)$ of the numbers 34086 and 14630. Write this as a linear combination of the numbers, i.e. find integers a and b such that $\gcd(34086, 14630) = 34086\,a + 14630\,b$.