THE MATRICULATION EXAMINATION BOARD — 16.3.2001 — MATHEMATICS ADVANCED COURSE

Ten problems at most may be answered. There are several items [marked **a)**, **b)** etc.] in some of the problems; all of them are to be considered for a complete answer.

1. Solve the equation

$$\frac{1}{x} - \frac{x}{x+3} = 0.$$

2. Find the equation for the line tangent to the curve $y = x^3$ at the point $(2, 8)$. At what point does this tangent line intersect the y-axis? Find the area of the triangle bounded by the tangent line, the y-axis and the line $y = 8$.

3. The vectors $\bar{a}$ and $\bar{b}$ have opposite directions. Let $\bar{a} = \frac{3}{2}\bar{\imath} - 2\bar{\jmath}$ and let the length of the vector $\bar{b}$ be 5. Find $\bar{b}$. What will be the end point if $\bar{b}$ is positioned so that it begins at the point $(4, 3)$?

4. A tank contains 2.3 kg air and a pump removes 5% of the air contents of the tank at each stroke. How many strokes are needed to have less than 0.2 kg air in the tank?

5. An icecream cornet has the shape of a right-circular cone. The height of the cornet is $h = 16$ cm and the diameter of the top (the base of the cone) is $d = 6$ cm. A protective paper in the shape of a circular sector is wrapped around the conical surface of the cone. Compute the radius r and the central angle α of the sector provided that the protective paper has no overlap (not even for glueing). Give the answers with the accuracy of one millimetre and one degree.

6. Find the maximum and minimum value of the function

$$f(x) = \frac{5}{4 + 3\cos 2x}$$

in the set of real numbers. What are the corresponding values of the argument?

7. In an investigation it was found that the average mass of a 200 gram packet of biscuits was 204 g and that the standard deviation was 6 g. Assume that the masses are normally distributed. Find the percentage of the packets whose mass was less than 200 g. Find the percentage of the packets with mass in the interval 200 g – 210 g.

8. All vertices of a right-angled triangle are on the parabola $y = x^2$; the tip of the right angle is at the vertex of the parabola. Show that the hypotenuse of every such triangle intersects the axis of the parabola at the same point. Find this point.

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9. The function f is defined in the interval $0 \leq x \leq 2$ as

$$f(x) = \int_0^1 |x - t| \, dt.$$

Find the maximum and minimum value of the function in the interval $[0, 2]$. Sketch the graph.

10. A function $f : \mathbb{R} \to \mathbb{R}$ is called *odd* if $f(-x) = -f(x)$ for each real number x. Give an example of an odd function which is increasing in $\mathbb{R}$, and of an odd function which is not increasing in $\mathbb{R}$. Show that $\lim_{x \to 0} f(x) = 0$, if f is odd and continuous.

11. A cube is placed inside a sphere so that each of its vertices is on the sphere. Another sphere is placed inside the cube so that it is tangent to each face of the cube. Further, a cube is placed inside the latter sphere in the same way etc. Show that the **a)** radii, **b)** surface-areas and **c)** volumes of the spheres each form a geometric sequence. Find the ratio of succesive terms for each of these sequences.

12. The base of a number system is 7. Express the numbers 11, 111 and 1111 of this system in the decimal system. Express the numbers 11, 111 and 1111 of the decimal system in this system.

13. The terms in a sequence are $x_1 = 1$, $x_2 = \sqrt{2}$, $x_3 = \sqrt{2\sqrt{2}}$, $x_4 = \sqrt{2\sqrt{2\sqrt{2}}}$ etc. Construct a recursive formula for the terms of the sequence. Compute $\lim_{n \to \infty} x_n$.

14. Give the definition for the conjugate $\overline{z}$ of the complex number $z = x + iy$. Show, using this definition, that the equation $\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2$ holds for the product of two complex numbers z_1 and z_2. Solve the equation $z^2 + \overline{z} + 1 = 0$.

15. At a fish-farm, a fish-disease spread in a salmon basin that originally held 1 100 fish. Owing to the effect of the disease the number of fish in the basin started to decrease according to the equation

$$P'(t) = -4\sqrt{P(t)}.$$

Here $P(t)$ is the number of fish at time t and the time t is measured in weeks. After how many weeks were all the fish dead?