THE MATRICULATION EXAMINATION BOARD — 1.10.1999 — MATHEMATICS ADVANCED COURSE

Questions 4, 7, 8 and 10 have alternatives, of which only one can be chosen.

1. A tree grows on level ground in a vertical position. The length of its shadow is 37 m when the angle of elevation of the sun is 28^o. Find the height of the tree.

2. Two balls are moulded from four decilitres of clay so that the volume of the smaller ball is one half of the volume of the larger one. Find the radiuses of the balls.

3. Solve the inequality $\frac{1}{5}x - 1 < \frac{1}{4}x + a$ where $a \in \mathbf{R}$ is a constant. Then examine for what values of a the set of solutions of the inequality is the interval $]5, \infty[$.

4. a) Find the real numbers x such that the derivative of the function $\cos(\frac{\pi}{2} - x)$ is $\sin x$.

   b) Suppose that a bank pays interest of p% per annum added to the account yearly. How much interest per annum added to the account half-yearly should the bank pay if the result were the same as in the first case?

5. The ratio of the two legs of a right-angled triangle is 1:2. Find the ratio where the height drawn to the hypotenuse divides the hypotenuse.

6. In a plane points $A = (-2, 0)$ and $B = (2, 3)$ are given. Find the set of points $P(x, y)$ in the plane satisfying the condition $|\overline{AP}| \leq 2|\overline{BP}|$. Sketch a figure.

7. a) It is written on a sack of flower seeds that the probability for a seed to germinate is 95% and that 5% of the contents of the bag are similar-looking weed seeds. The seeds of the sack are divided into bags containing 20 seeds. Find the probability that a gardener who sows the seeds from such a bag gets at least 19 plants of flowers that he wants. Find also the probability that he sows at least one weed seed.

   b) Show that the function $f : \mathbf{R}^2 \to \mathbf{R}, f(x, y) = x^2 - 3xy + 3y^2$ of two real variables has only non-negative values. Show also that the function attains all non-negative real values.

8. a) Places A and B are situated in the northern hemisphere so that both have the degree of latitude of 49 but the difference in the degrees of longitude is 38. Find the distance from A to B measured along the latitude. Find also the shortest distance between A and B measured

along the surface of the earth. (The unevenness of the surface can be disregarded.) The length of the great circle of the earth is 40 000 km here. Give the answer to three significant digits.

b) Two vertices of a triangle are at points (1,3) and (2,5). The third vertex lies on the curve $y = \ln(1+x)$. Find the co-ordinates of the third vertex provided that the area of the triangle is as small as possible.

9. A function $f : [0,\infty[\to \mathbf{R}$ is defined as follows: $f(x) = 2^{-n}(-x+n)(x-n-1)$, for $n \leq x < n+1,\ n = 0,1,2,3,\ldots$. Find the area that lies between the curve and the x-axis.

10. a) How is the derivative of a function f defined at a point $x = a$? Find an example of a function which does not have a derivative at point $x = 1$. A function $f : \mathbf{R} \to \mathbf{R}$ is defined as follows:

$$f(x) = \begin{cases} x^2 \sin\dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Examine whether f has a derivative at point $x = 0$.

b) The graph of a positive, differentiable and increasing function f passes through the point (0,1). Let T be the tangent line at an arbitrary point $(x_o, f(x_o))$ of its graph. The line $x = x_o$, the x-axis and T bound a triangle with the area $A = f(x_o)$. Find the function f.